



WESTFÄLISCHE
WILHELMS-UNIVERSITÄT
MÜNSTER

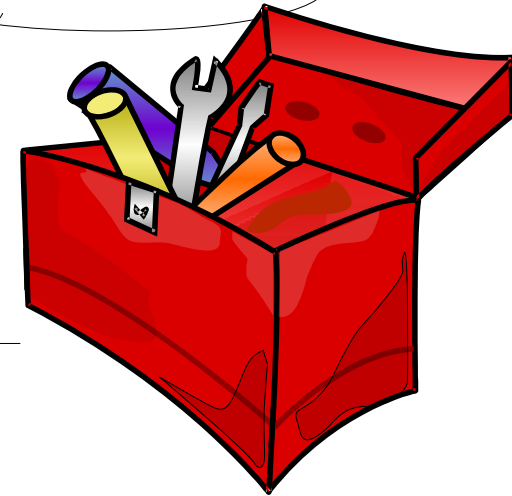
Software concepts for coupling reduced and detailed simulations

Workshop on Reduced Basis Methods, Ulm

> Motivation

PDE Discretizations (FEM, FV, DG):
Navier-Stokes, Groundwater-Flow, Convection-Diffusion,
Poisson, Maxwell, ...

Parametrization



Magical RB-Tools

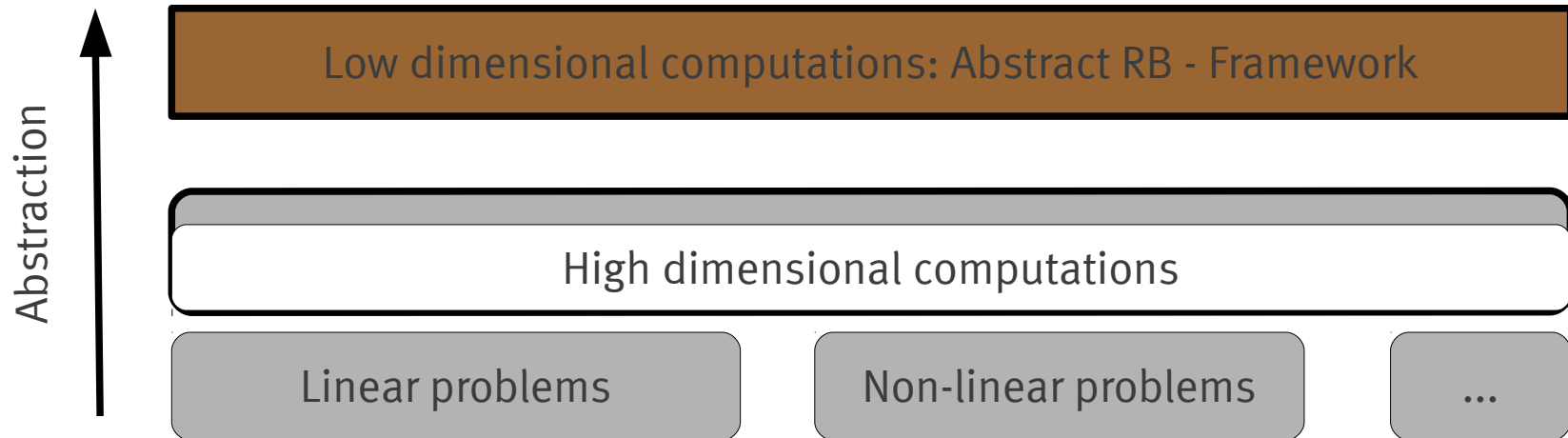
Reduced model



› Goals Specification

- Abstract Reduced Basis Framework
- Re-use of (existing) efficient implementations of PDE discretizations
- Short time for implementing the detailed simulations as reduced simulations

› Abstract Software Concept



PDE Discretizations (FEM, FV, DG):
Navier-Stokes, Groundwater-Flow, Convection-Diffusion,
Poisson, Maxwell, ...

› Low dimensional Computations (RBMATLAB)

- Control of Offline-Algorithms
 - POD-Greedy Algorithm, Empirical Interpolation Algorithm
 - Gathering and post-processing of reduced matrices
- Low-Dimensional Computations
 - Reduced Simulations
 - A posteriori error estimation
- Low dimensional data visualization
- (Implemented in MATLAB)

High dimensional computations ()

- Storage / Manipulation of reduced spaces
- Efficient high dimensional linear algebra algorithms
 - POD, Orthonormalization, Gram-Matrix computations
- Parametrization
- (Implemented in C++, based on
 - DUNE core modules (<http://dune-project.org>) and
 - DUNE-FEM (dune.mathematik.uni-freiburg.de)

The Glue

- Communication between DUNE-RB and RBMATLAB can be realized by
 1. compiling DUNE-RB as (mex-) library for matlab or:
 2. TCP/IP Communication between two stand-alone applications





Proof of Concept (linear evolution problem)

RBMATLAB > matlab

```
< M A T L A B (R) >  
Copyright 1984-2008 The MathWorks, Inc.  
Version 7.6.0.324 (R2008a)  
February 10, 2008
```

To get started, type one of these: helpwin, helpdesk, or demo

For product information, visit www.mathworks.com.

```
starting up rbmatlab in directory:  
/home/martin/projects/rbm-results/rbmatlab  
Using the following directory for large temporary data:  
/tmp  
Using the following directory for data files storing results:  
/home/martin/projects/rbm-results/results  
Using the following directory as RBMATLABHOME:  
/home/martin/projects/rbm-results/rbmatlab  
skipped clearing filecache for function-calls!  
>> 
```

DUNE-RB > ./dunerbServer

```
YaspGridParameterBlock: Parameter 'overlap' not specified, defaulting to '0'.  
server: waiting for connections...
```




Proof of Concept (linear evolution problem)

```
< M A T L A B (R) >  
Copyright 1984-2008 The MathWorks, Inc.  
Version 7.6.0.324 (R2008a)  
February 10, 2008
```

To get started, type one of these: helpwin, helpdesk, or demo
.
For product information, visit www.mathworks.com.

```
starting up rbmatlab in directory:  
/home/martin/projects/rbm-results/rbmatlab  
Using the following directory for large temporary data:  
/tmp  
Using the following directory for data files storing results:  
/home/martin/projects/rbm-results/results  
Using the following directory as RBMATLABHOME:  
/home/martin/projects/rbm-results/rbmatlab  
skipped clearing filecache for function-calls!  
>>
```

```
>> [a,b,c] = ...  
mexclient('echo', [1, 2], ...  
          struct('field', [1,2]), ...  
          { [1,2], [3,4] });
```

```
client: connect: Connection refused  
Warning: connection to ::1 failed
```

```
client connected to 127.0.0.1  
copying argument no. 0  
copying argument no. 1  
copying argument no. 2  
>> □
```

```
DUNE-RB > ./dunerbServer  
YaspGridParameterBlock: Parameter 'overlap' not specified, defaulting to '0'.  
server: waiting for connections...  
server: got connection from 127.0.0.1
```

```
Received call for processing 'echo'  
with 4 arguments and 3 return values.  
=====
```

```
copying argument no. 0  
copying argument no. 1  
copying argument no. 2
```



Proof of Concept (linear evolution problem)

```
/home/martin/projects/rbm-results/rbmatlab
skipped clearing filecache for function-calls!
>>
>> [a,b,c] = ...
    mexclient('echo', [1, 2], ....
                struct('field', [1,2]), ...
                { [1,2], [3,4] });
client: connect: Connection refused
Warning: connection to ::1 failed
```

```
client connected to 127.0.0.1
copying argument no. 0
copying argument no. 1
copying argument no. 2
```

```
>> a
a =
    1    2

>> b
b =
    field: [1 2]

>> c
c =
    [1x2 double]    [1x2 double]
```

```
>> █
```

```
DUNE-RB > ./dunerbServer
YaspGridParameterBlock: Parameter 'overlap' not specified, defaulting to '0'.
server: waiting for connections...
server: got connection from 127.0.0.1

Received call for processing 'echo'
with 4 arguments and 3 return values.
=====

copying argument no. 0
copying argument no. 1
copying argument no. 2
█
```

Proof of Concept (linear evolution problem)

< M A T L A B (R) >
Copyright 1984-2008 The MathWorks, Inc.
Version 7.6.0.324 (R2008a)
February 10, 2008

To get started, type one of these: helpwin, helpdesk, or demo

For product information, visit www.mathworks.com.

```
starting up rbmatlab in directory:
/home/martin/projects/rbm-results/rbmatlab
Using the following directory for large temporary data:
/tmp
Using the following directory for data files storing results:
/home/martin/projects/rbm-results/results
Using the following directory as RBMATLABHOME:
/home/martin/projects/rbm-results/rbmatlab
skipped clearing filecache for function-calls!
>>
```

```
>> % load model parameters
>>
>> model = convdiff_dune_model;
```

Warning: Name is nonexistent or not a directory: mexclient.

> In path at 110

In addpath at 87

In convdiff_dune_model at 95

client: connect: Connection refused

Warning: connection to ::1 failed

client connected to 127.0.0.1

```
>> []
```

```
DUNE-RB > ./dunerbServer
YaspGridParameterBlock: Parameter 'overlap' not specified, defaulting to '0'.
server: waiting for connections...
server: got connection from 127.0.0.1
```

```
Received call for processing 'init_model'
with 1 arguments and 1 return values.
=====
```

```
read discfunclist_xdr from headerfile, size = 20
Using the explicit ode solver! In order to use a different discretization, change the 'DISCRETIZATION' make variable
```

```
Received call for processing 'get_mu'
with 1 arguments and 1 return values.
=====
```

```
Received call for processing 'rb_symbolic_coefficients'
with 1 arguments and 1 return values.
=====
```

Proof of Concept (linear evolution problem)

```
> In path at 110
  In addpath at 87
  In convdiff_dune_model at 95
client: connect: Connection refused
Warning: connection to ::1 failed
```

```
client connected to 127.0.0.1
```

```
>>
```

```
>> model.rb_problem_type
```

```
ans =
```

```
lin_evol
```

```
>> model.RB_generation_mode
```

```
ans =
```

```
greedy_uniform_fixed
```

```
>> model.RB_stop_Nmax
```

```
ans =
```

```
20
```

```
>> model.T % this is read from DUNE-RB <<<<<<
```

```
ans =
```

```
1
```

```
>> █
```

```
DUNE-RB > ./dunerbServer
```

```
YaspGridParameterBlock: Parameter 'overlap' not specified, defaulting to '0'.
```

```
server: waiting for connections...
```

```
server: got connection from 127.0.0.1
```

```
Received call for processing 'init_model'
with 1 arguments and 1 return values.
```

```
=====

read discfunclist_xdr from headerfile, size = 20
Using the explicit ode solver! In order to use a different discretization, change the 'DISCRETIZATION' make variable
```

```
Received call for processing 'get_mu'
with 1 arguments and 1 return values.
```

```
=====

Received call for processing 'rb_symbolic_coefficients'
with 1 arguments and 1 return values.
```

```
=====

█
```

Proof of Concept (linear evolution problem)

```
ans =  
  
lin_evol  
  
>> model.RB_generation_mode  
  
ans =  
  
greedy_uniform_fixed  
  
>> model.RB_stop_Nmax  
  
ans =  
  
    20  
  
>> model.T           % this is read from DUNE-RB <<<<<<  
  
ans =  
  
    1  
  
>>  
>>  
>>  
>>  
>> % generate high dimensional model specific data, like e.g.  
>> % the grid  
>>  
>> model_data = gen_model_data(model);  
>>  
>> █
```

```
DUNE-RB > ./dunerbServer  
YaspGridParameterBlock: Parameter 'overlap' not specified, defaulting to '0'.  
server: waiting for connections...  
server: got connection from 127.0.0.1  
  
Received call for processing 'init_model'  
with 1 arguments and 1 return values.  
=====
```

read discfunclist_xdr from headerfile, size = 20
Using the explicit ode solver! In order to use a different discretization, change the 'DISCRETIZATION' make variable

```
Received call for processing 'get_mu'  
with 1 arguments and 1 return values.  
=====
```

```
Received call for processing 'rb_symbolic_coefficients'  
with 1 arguments and 1 return values.  
=====
```

```
Received call for processing 'gen_model_data'  
with 1 arguments and 1 return values.  
=====
```

█

Proof of Concept (linear evolution problem)

```
>> % Just for fun: Do a DETAILED simulation in DUNE-RB >>>>>>
>>
```

```
>> % first set the parameter mu
>> model = set_mu(model, [0.0 0.5 1.0]);
>>
>> % then run the simulation
>> tic; sim_data = detailed_simulation(model, model_data); toc
Elapsed time is 11.795319 seconds.
```

```
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
```

```
Received call for processing 'rb_symbolic_coefficients'
with 1 arguments and 1 return values.
=====
```

```
Received call for processing 'gen_model_data'
with 1 arguments and 1 return values.
=====
```

```
Received call for processing 'set_mu'
with 2 arguments and 0 return values.
=====
```

```
Received call for processing 'detailed_simulation'
with 1 arguments and 1 return values.
=====
```

```
opening file: ./grape//solution.series
```

```
Received call for processing 'set_mu'
with 2 arguments and 0 return values.
=====
```

```
Received call for processing 'detailed_simulation'
with 1 arguments and 1 return values.
=====
```

```
opening file: ./grape//solution.series
```

Proof of Concept (linear evolution problem)

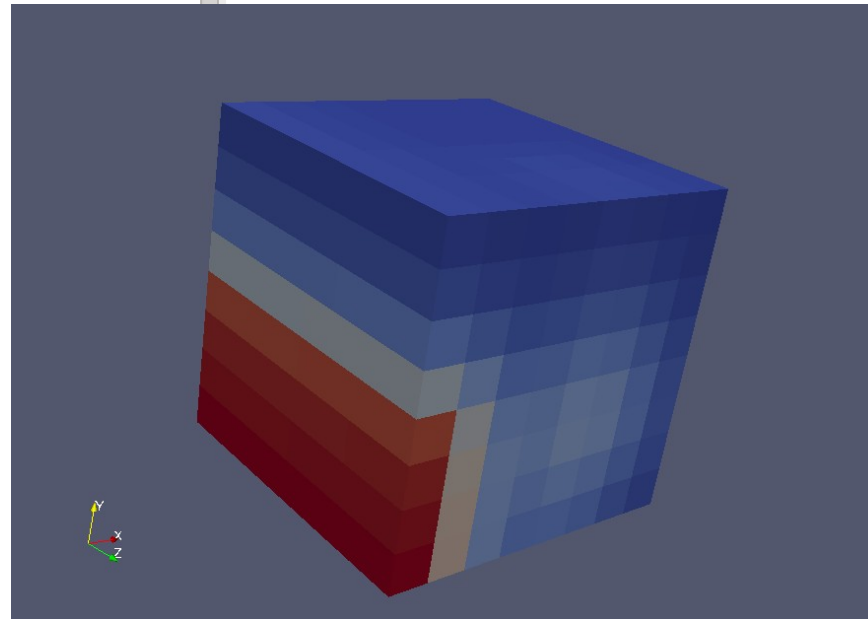
```
>> % Just for fun: Do a DETAILED simulation in DUNE-RB >>>>>>
```

```
>> % first set the parameter mu
>> model = set_mu(model, [0.0 0.5 1.0]);
>> % then run the simulation
>> tic; sim_data = detailed_simulation(model, model_data); toc
Elapsed time is 11.795319 seconds.
```

11.8 seconds!

```
Received call for processing 'rb_symbolic_coefficients'
with 1 arguments and 1 return values.
=====
```

```
Received call for processing 'gen_model_data'
with 1 arguments and 1 return values.
=====
```



```
opening file: ../grape//solution.series
```

Proof of Concept (linear evolution problem)

```
>> % Generate the reduced basis with the POD-Greedy algorithm
>> % in DUNE-RB >>>>>>>
>>
>> detailed_data = gen_detailed_data(model, model_data);
Starting RB extension loop
```

```
Detected maximum error prediction 0.044006 for mu=[0.001
1 0.5]
Extended RB to length 2
```

```
Detected maximum error prediction 0.015456 for mu=[0 1
Extended RB to length 3
```

```
Detected maximum error prediction 0.012877 for mu=[0
1 0.5]
Extended RB to length 4
```

```
Detected maximum error prediction 0.01064 for mu=[0
0.5]
Extended RB to length 5
```

```
Detected maximum error prediction 0.0084073 for mu=[0.001
0.5 1]
Extended RB to length 6
```

```
Detected maximum error prediction 0.0073233 for mu=[0 1 1]
Extended RB to length 7
```

```
Detected maximum error prediction 0.0055443 for mu=[0 1 1]
Extended RB to length 8
```

```
Detected maximum error prediction 0.0048443 for mu=[0
1 0.5]
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'get_mu'
with 1 arguments and 1 return values.
=====
```

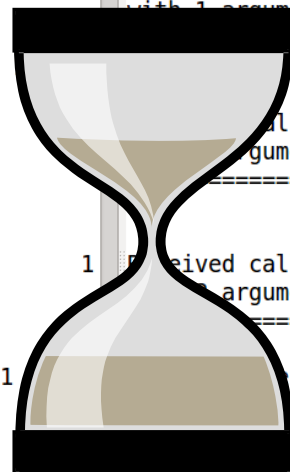
```
Received call for processing 'set_mu'
with 1 arguments and 0 return values.
=====
```

```
1 Received call for processing 'rb_extension_PCA'
with 2 arguments and 1 return values.
=====
```

```
./grape//solution.series
```

```
Received call for processing 'set_mu'
with 1 arguments and 0 return values.
=====
```

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```



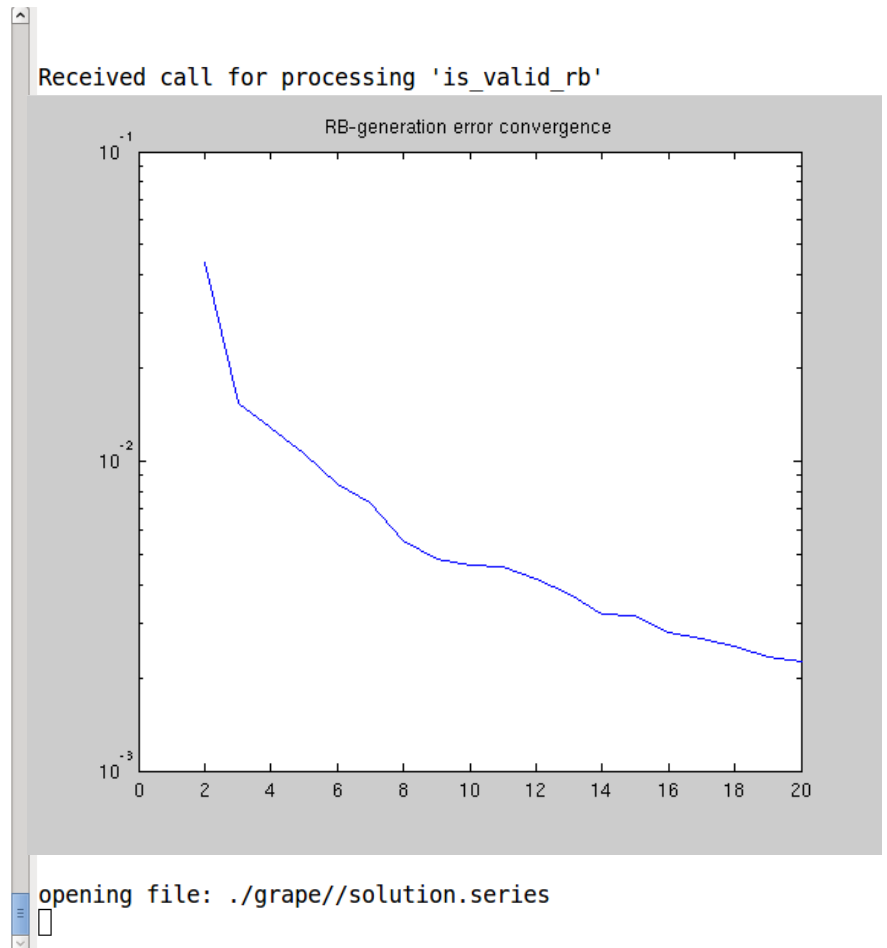
Proof of Concept (linear evolution problem)

```
>>
>>
>> detailed_data.RB_info

ans =

      M_train: [3x64 double]
    max_err_sequence: [1x20 double]
      mu_sequence: [3x20 double]
      mu_ind_seq: [1x20 double]
    toc_value_sequence: [1x20 double]
      M_first_errs: [64x1 double]
    stopped_on_epsilon: 0
stopped_on_max_val_train_ratio: 0
    stopped_on_timeout: 0
      stopped_on_Nmax: 1
    stopped_on_empty_extension: 0
    stopped_on_Nlimit_estimation: 0
      M_last_errs: [64x1 double]
    elapsed_time: 3.0000e-06
```

```
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
```



Proof of Concept (linear evolution problem)

```
>> % NOW: Generate reduced matrices for online computations
>> % and get them to RBMATLAB <<<<<<<<
```

```
>>
>> reduced_data = gen_reduced_data(model, detailed_data)
```

```
reduced_data =
```

```
    a0: {[1x20 double]}
    LL_I: {2x1 cell}
    LL_E: {5x1 cell}
        bb: {4x1 cell}
    K_II: {4x1 cell}
    K_IE: {10x1 cell}
    K_EE: {25x1 cell}
        m_I: {8x1 cell}
        m_E: {20x1 cell}
        m: {16x1 cell}
    N: 20
```

```
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
>>
```

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'set_mu'
with 2 arguments and 0 return values.
=====
```

```
Received call for processing 'reconstruct_and_compare'
with 2 arguments and 0 return values.
=====
```

```
opening file: ./grape//solution.series
```

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

Proof of Concept (linear evolution problem)

```
>> % The matrices are all of small sizes: e.g. the explicit
>> % discretization operator: LL_E
>>
>> sizes=cellfun(@(X) size(X), reduced_data.LL_E, ...
                'UniformOutput', false);
>> sizes{:}
```

ans =

```
20    20
```

ans =

```
20    20
```

ans =

```
20    20
```

ans =

```
20    20
```

ans =

```
20    20
```

```
>>
```

```
>> █
```

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'set_mu'
with 2 arguments and 0 return values.
=====
```

```
Received call for processing 'reconstruct_and_compare'
with 2 arguments and 0 return values.
=====
```

```
opening file: ./grape//solution.series
```

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

Proof of Concept (linear evolution problem)

```
>> % Now fast reduced simulations are possible in RBMATLAB
>> % without any communication to DUNE-RB
>>
>> model = model.set_mu(model, [0 0.5 1], true);
>>
>> tic; rb_sim_data=rb_simulation(model, reduced_data); toc
Elapsed time is 0.020223 seconds.
```

```
>>
>>
>> rb_sim_data
```

```
rb_sim_data =
```

```
    a: [20x113 double]
  Delta: [1x113 double]
  LL_I: [20x20 double]
  LL_E: [20x20 double]
```

```
>>
>>
```

```
>> % Error estimator at end time:
>>
>> rb_sim_data.Delta(end)
```

```
ans =
```

```
    0.0019
```

```
>>
>>
>>
>>
>>
```

0.02 seconds!

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'set_mu'
with 2 arguments and 0 return values.
=====
```

```
Received call for processing 'reconstruct_and_compare'
with 2 arguments and 0 return values.
=====
```

```
opening file: ./grape//solution.series
```

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

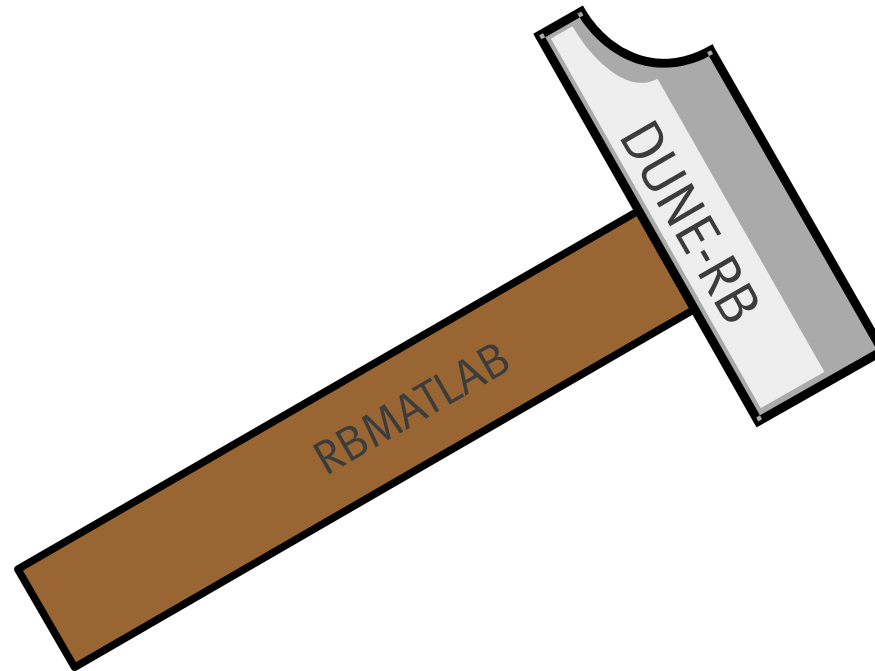

Linear Transport Problem

	High-dim. Solution (sec)	RB Gen- eration	Gen. of Online Matrices	Reduced Sim.	Recon- struction	Grid Cells	$L^\infty - L^2$ - Error ¹
2D Transport (25 Base Functions)	11	71	6.69	0.11	0.33	1,024	$1.42 \cdot 10^{-3}$
2D Transport (50 Base Functions)	11	2,250	21	0.15	0.42	1,024	$4.64 \cdot 10^{-4}$
3D Transport (50 Base Functions)	944	$1.57 \cdot 10^5$	4,659	0.15	26	32,768	$9.11 \cdot 10^{-4}$

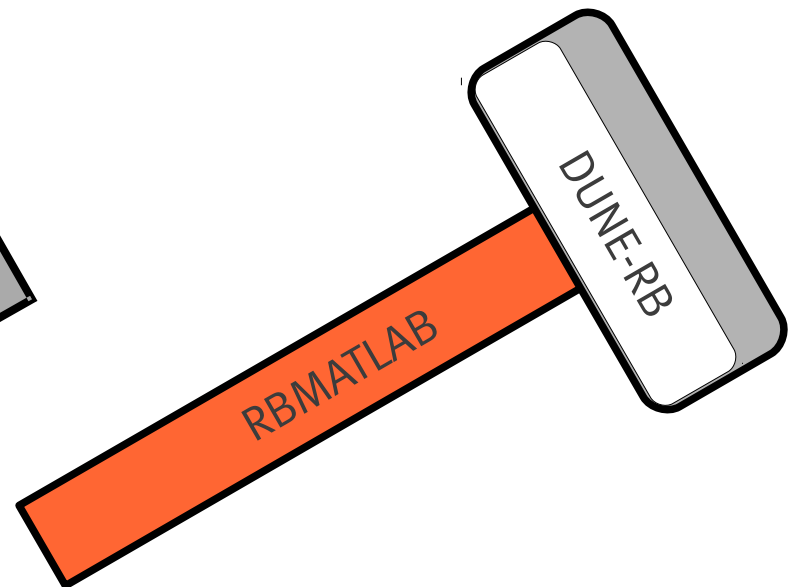
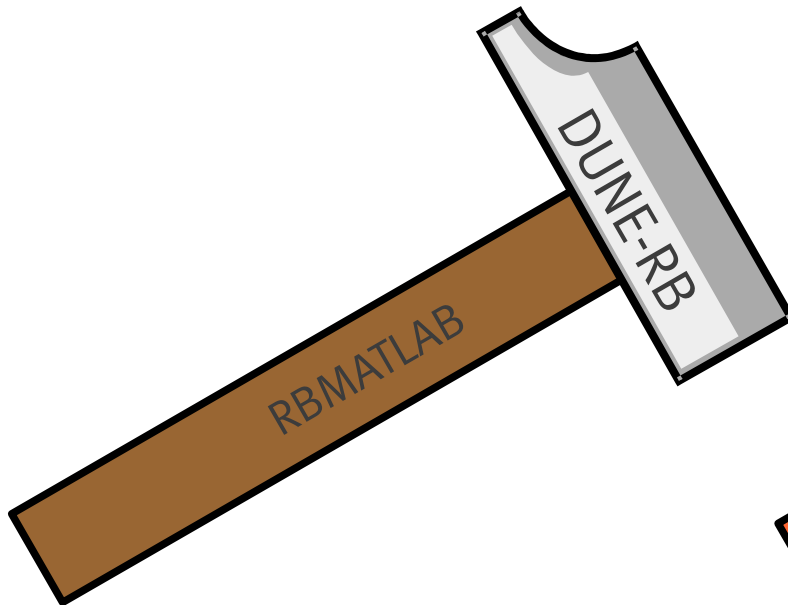
Table 1: Numerical results for a transport problem in 2D and 3D with non-divergence-free velocity.



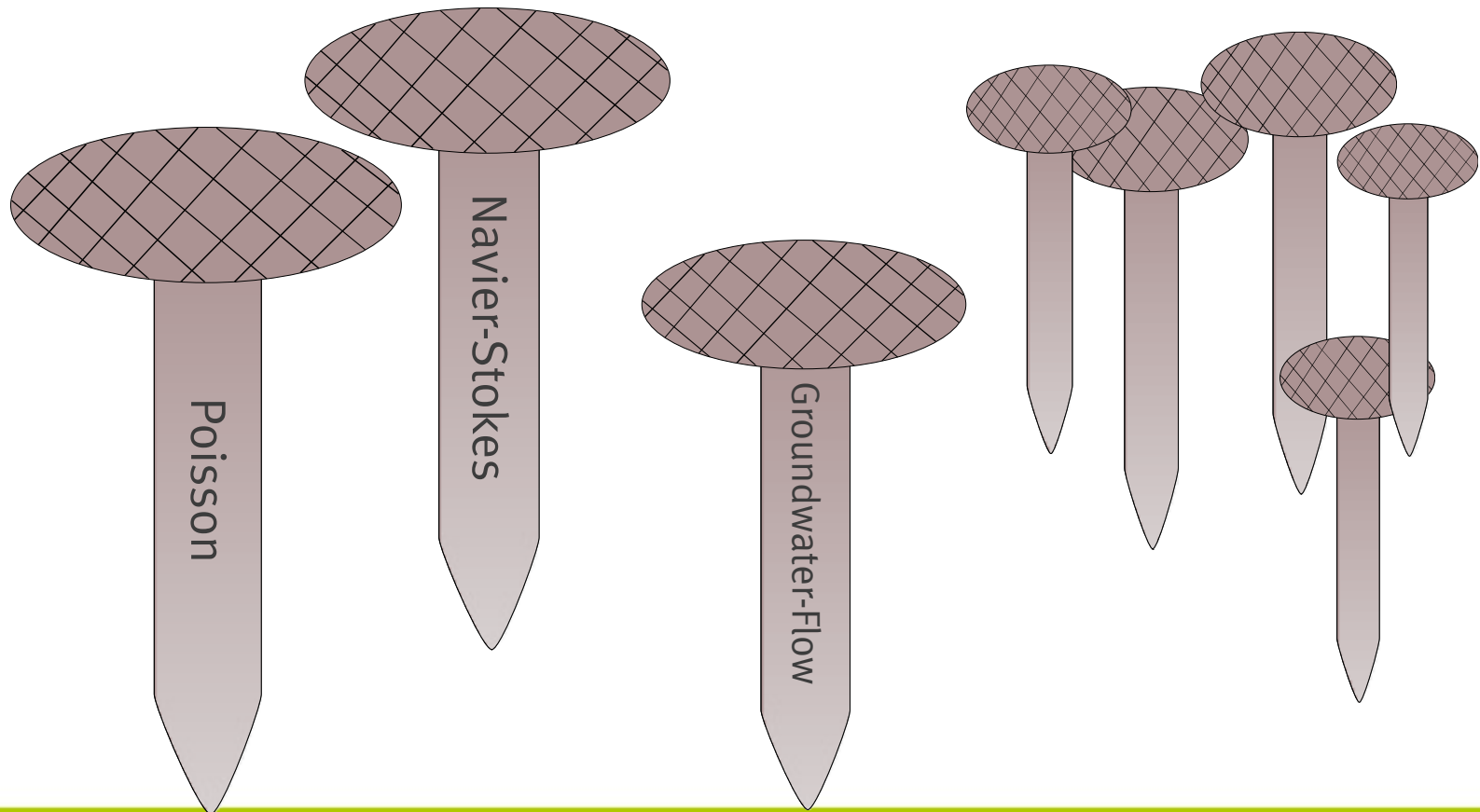
So, we have a hammer for linear evolution problems ...



... and even one for non-linear problems ...



But how do you make your problems look like a nail?



Interface to PDE descretizations

```
>> % Generate the reduced basis with the POD-Greedy algorithm
>> % in DUNE-RB >>>>>>>>
>>
>> detailed_data = gen_detailed_data(model, model_data);
Starting RB extension loop
```

```
Detected maximum error prediction 0.044006 for mu=[0.001
1 0.5]
Extended RB to length 2
```

```
Detected maximum error prediction 0.015456 for mu=[0 1 1]
Extended RB to length 3
```

```
Detected maximum error prediction 0.012877 for mu=[0
1 0.5]
Extended RB to length 4
```

```
Detected maximum error prediction 0.01064 for mu=[0
0.5]
Extended RB to length 5
```

```
Detected maximum error prediction 0.0084073 for mu=[0.001
0.5 1]
Extended RB to length 6
```

```
Detected maximum error prediction 0.0073233 for mu=[0 1 1]
Extended RB to length 7
```

```
Detected maximum error prediction 0.0055443 for mu=[0 1 1]
Extended RB to length 8
```

```
Detected maximum error prediction 0.0048443 for mu=[0
1 0.5]
```

```
Received call for processing 'rb_operators'
with 2 arguments and 1 return values.
=====
```

```
Received call for processing 'get_mu'
with 1 arguments and 1 return values.
=====
```

```
Received call for processing 'set_mu'
with 2 arguments and 0 return values.
=====
```

```
1 Received call for processing 'rb_extension_PCA'
with 3 arguments and 1 return values.
=====
```

```
opening file: ./grape//solution.series
```

```
Received call for processing 'set_mu'
with 2 arguments and 0 return values.
=====
```

```
Received call for processing 'rb_init_values'
with 2 arguments and 1 return values.
=====
```

Interface to PDE discretizations (linear)

- For PDE discretization of form

For $\mu \in \mathcal{P}$ compute $\{u_h^k(\mu)\}_{k=0}^K \subset \mathcal{W}_h \subset L^2(\Omega)$ by

$$u_h^0(\mu) := P[u_0(\mu)]$$

$$u_h^k(\mu) - \Delta t \mathcal{L}^I(\mu)[u_h^k(\mu)] := u_h^{k-1}(\mu) + \Delta t \mathcal{L}^E(\mu)[u_h^{k-1}(\mu)].$$

- Implement affinely parameter dependent operators

$$\mathcal{L} = \sum_{q=1}^{Q_L} \boxed{\sigma_L^q(\mu)} \boxed{\mathcal{L}^q}$$

Interface to PDE discretizations (linear)

Affinely parametrized operators

$$\mathcal{L} = \sum_{q=1}^{Q_L} \boxed{\sigma_L^q(\mu)} \boxed{\mathcal{L}^q}$$

with methods

- Parametrization `[set_mu()]`
- Parameter dependent **coefficients** `[coefficient()]`
- Parameter independent **operators** `[component()]`

Interface to PDE discretizations (linear)

Affinely parametrized operators

$$\mathcal{L} = \sum_{q=1}^{Q_L} \boxed{\sigma_L^q(\mu)} \boxed{\mathcal{L}^q}$$

with methods

- Parametrization `[set_mu()]`
- Parameter dependent **coefficients** `[coefficient()]`
- Parameter independent **operators** `[component()]`

⇒ Automatic generation of reduced matrices:

$$(\mathbf{L}^q)_{n,m} = \int_{\Omega} \mathcal{L}^q[\varphi_m] \varphi_n, \quad q = 1, \dots, Q_L, m, n = 1, \dots, N$$

Interface to PDE discretizations (non-linear)

- For PDE discretizations of the form

For $\mu \in \mathcal{P}$ find $\{u_h\}_{k=0}^K \subset \mathcal{W}_h \subset L^2(\Omega)$, s.t.

$$u_h^0 := P_h[u_0(\mu)], \quad u_h^{k+1} := u_h^{k+1, \nu_{\max}(k)}$$

with Newton iteration

$$u_h^{k+1,0} := u_h^k, \quad u_h^{k+1,\nu+1} := u_h^{k+1,\nu} + \delta_h^{k+1,\nu+1},$$

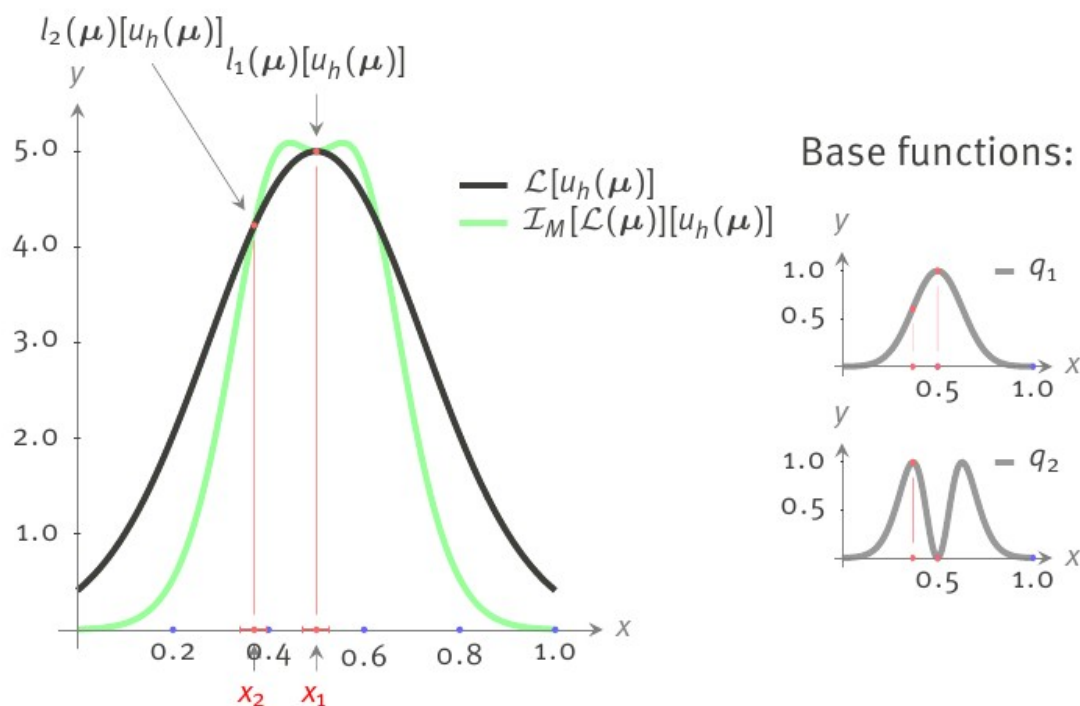
$$\left(\text{Id} + \Delta t \mathbf{D} \mathcal{L}_h^l|_{u_h^{k+1,\nu}} \right) [\delta_h^{k+1,\nu+1}] = u_h^k - u_h^{k+1,\nu} - \Delta t \left(\mathcal{L}_h^l[u_h^{k+1,\nu}] + \mathcal{L}_h^E[u_h^k] \right).$$

- Implement empirical interpolation for discrete operators

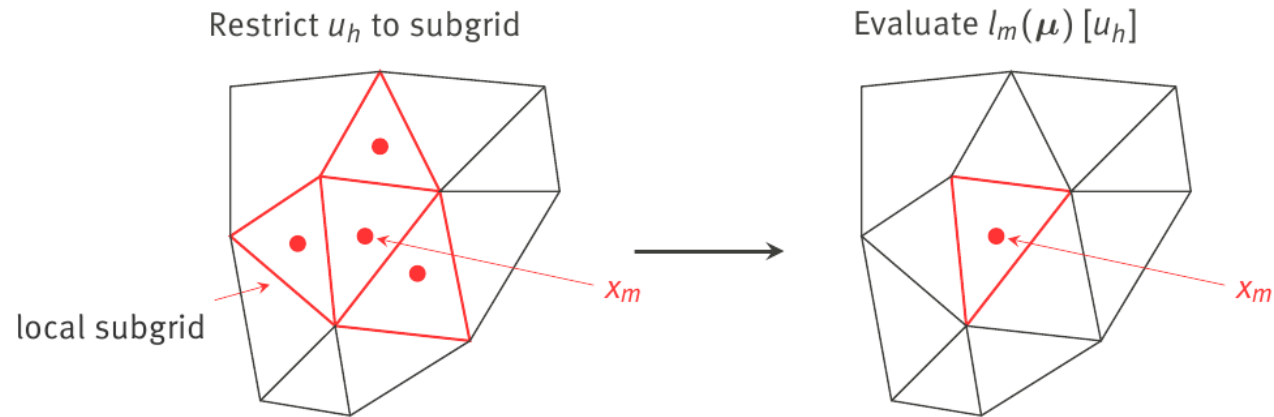
$$\mathcal{L}_h(\mu)[u_h^k(\mu)] \approx \mathcal{I}_M[\mathcal{L}_h(\mu)][u_h^k(\mu)] = \sum_{m=1}^M l_m(\mu)[u_h^k(\mu)] \xi_m$$

The coefficient functionals $l_m(\mu) = \mathcal{L}_h(\mu)[\cdot](x_m)$ can be computed efficiently, if

- Operator has localized structure (small stencil) and
- Local geometry information can be precomputed in offline phase.

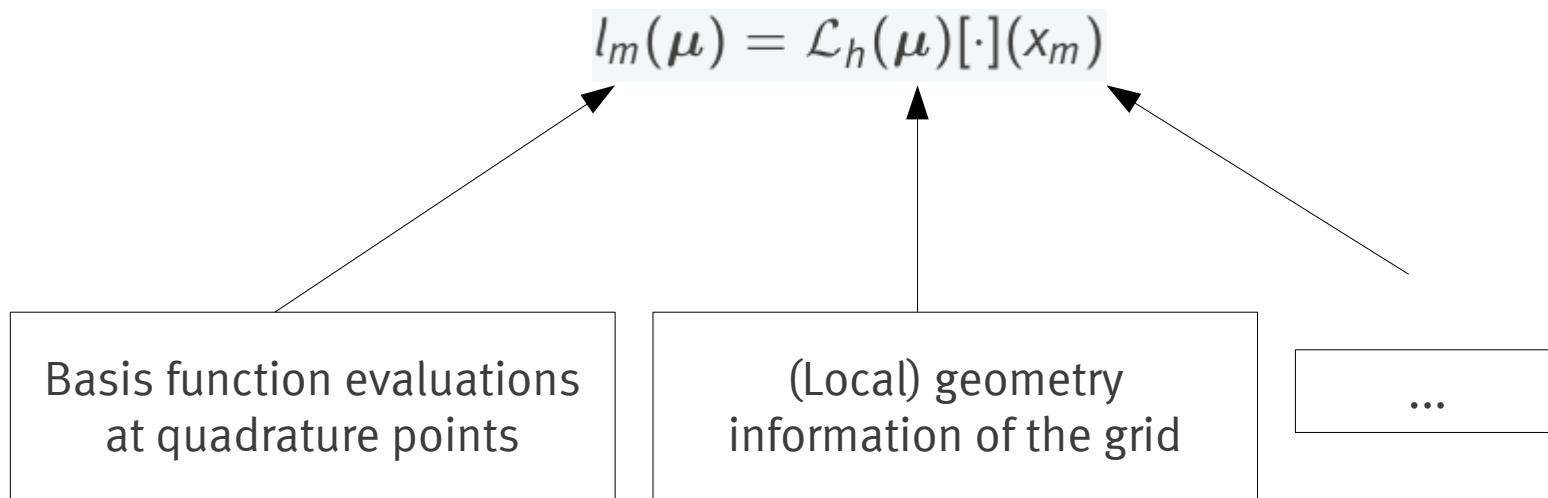


Empirical Interpolation for Operators (local grid)



Automatic generation of localized operators

Usual dependencies for local operator evaluations:



Automatic generation of localized operators

Therefore: Delegation of function calls concerning grid geometry and discrete function space to a wrapper:

$$l_m(\mu) = \mathcal{L}_h(\mu)[\cdot](x_m)$$

DUNE-RB Grid Wrapper

Basis function evaluations
at quadrature points

(Local) geometry
information of the grid

...

Behaviour of DUNE-RB Grid Wrapper

- During detailed simulation:

Delegate calls **directly** to the grid and the discrete function space

- During offline phase:

store all grid and discrete function space information concerning the **subgrid** in **low-dimensional** data structures

- During online phase:

Delegate calls to **low-dimensional** data structures



Conclusion

- Generic implementation of reduced base tools for linear problems is possible
- Empirical interpolation for non-linear problems can be realized with subgrid extraction
- Future work:
 - More examples
 - Tarball – Release with installation instructions is planned, and will be published on <http://morepas.org>.



WESTFÄLISCHE
WILHELMS-UNIVERSITÄT
MÜNSTER

Thank you for your attention!

wissen.leben
WWU Münster